

# SMALL FURSTENBERG SETS

URSULA MOLTER AND EZEQUIEL RELA

**ABSTRACT.** For  $\alpha$  in  $(0, 1]$ , a subset  $E$  of  $\mathbb{R}^2$  is called *Furstenberg set* of type  $\alpha$  or  $F_\alpha$ -set if for each direction  $e$  in the unit circle there is a line segment  $\ell_e$  in the direction of  $e$  such that the Hausdorff dimension of the set  $E \cap \ell_e$  is greater or equal than  $\alpha$ . In this paper we show that if  $\alpha > 0$ , there exists a set  $E \in F_\alpha$  such that  $\mathcal{H}^g(E) = 0$  for  $g(x) = x^{\frac{1}{2} + \frac{3}{2}\alpha} \log^{-\theta}(\frac{1}{x})$ ,  $\theta > \frac{1+3\alpha}{2}$ , which improves on the previously known bound, that  $H^\beta(E) = 0$  for  $\beta > \frac{1}{2} + \frac{3}{2}\alpha$ . Further, by refining the argument in a subtle way, we are able to obtain a sharp dimension estimate for a whole class of *zero-dimensional* Furstenberg type sets. Namely, for  $\mathfrak{h}_\gamma(x) = \log^{-\gamma}(\frac{1}{x})$ ,  $\gamma > 0$ , we construct a set  $E_\gamma \in F_{\mathfrak{h}_\gamma}$  of Hausdorff dimension not greater than  $\frac{1}{2}$ . Since in a previous work we showed that  $\frac{1}{2}$  is a lower bound for the Hausdorff dimension of any  $E \in F_{\mathfrak{h}_\gamma}$ , with the present construction, the value  $\frac{1}{2}$  is sharp for the whole class of Furstenberg sets associated to the zero dimensional functions  $\mathfrak{h}_\gamma$ .

## 1. INTRODUCTION

We study dimension properties of sets of Furstenberg type. In particular we are interested to be able to construct very small Furstenberg sets in a given class. The notion of size of a set is its Hausdorff dimension, denoted by  $\dim_H$ . We begin with the definition of classical Furstenberg sets.

**Definition 1.1.** For  $\alpha$  in  $(0, 1]$ , a subset  $E$  of  $\mathbb{R}^2$  is called *Furstenberg set* of type  $\alpha$  or  $F_\alpha$ -set if for each direction  $e$  in the unit circle there is a line segment  $\ell_e$  in the direction of  $e$  such that the Hausdorff dimension of the set  $E \cap \ell_e$  is greater or equal than  $\alpha$ . We will also say that such set  $E$  belongs to the class  $F_\alpha$ .

It is known ([Wol99], [KT01], [MR10]) that  $\dim_H(E) \geq \max\{2\alpha, \alpha + \frac{1}{2}\}$  for any  $F_\alpha$ -set  $E \subseteq \mathbb{R}^2$  and there are examples of  $F_\alpha$ -sets  $E$  with  $\dim_H(E) \leq \frac{1}{2} + \frac{3}{2}\alpha$ . Hence, if we denote by

$$\gamma(\alpha) = \inf\{\dim_H(E) : E \in F_\alpha\},$$

then

$$(1) \quad \max\left\{\alpha + \frac{1}{2}; 2\alpha\right\} \leq \gamma(\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \quad \alpha \in (0, 1].$$

---

1991 *Mathematics Subject Classification.* Primary 28A78, 28A80.

*Key words and phrases.* Furstenberg sets, Hausdorff dimension, dimension function, Jarník's theorems.

This research is partially supported by Grants: PICT2006-00177, UBACyT X149 and CONICET PIP368.

In [MR10] the left hand side of this inequality has been extended to the case of more general dimension functions, i.e., functions that are not necessarily power functions ([Hau18]).

### 1.1. Dimension Functions and Hausdorff measures.

**Definition 1.2.** A function  $h$  will be called *dimension function* if it belongs to the following class:

$$\mathbb{H} := \{h : [0, \infty) \rightarrow [0 : \infty), \text{non-decreasing, right continuous, } h(0) = 0\}.$$

If one looks at the power functions, there is a natural total order given by the exponents. If we denote by  $h_\alpha(x) = x^\alpha$ , then  $h_\alpha$  is *dimensionally smaller* than  $h_\beta$  if and only if  $\alpha < \beta$ . In  $\mathbb{H}$ , however, by extending this natural notion of order we only obtain a *partial* order.

**Definition 1.3.** Let  $g, h$  be two dimension functions. We will say that  $g$  is dimensionally smaller than  $h$  and write  $g \prec h$  if and only if

$$\lim_{x \rightarrow 0^+} \frac{h(x)}{g(x)} = 0.$$

We will be particularly interested in the special subclass of dimension functions that allows us to classify zero dimensional sets.

**Definition 1.4.** A function  $h \in \mathbb{H}$  will be called “zero dimensional dimension function” if  $h \prec x^\alpha$  for any  $\alpha > 0$ . We denote by  $\mathbb{H}_0$  the subclass of those functions.

As usual, the  $h$ -dimensional (outer) Hausdorff measure  $\mathcal{H}^h$  will be defined as follows. For a set  $E \subseteq \mathbb{R}^n$  and  $\delta > 0$ , write

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_i h(\text{diam}(E_i)) : E \subset \bigcup_i E_i, \text{diam}(E_i) < \delta \right\}.$$

The  $h$ -dimensional Hausdorff measure  $\mathcal{H}^h$  of  $E$  is defined by

$$(2) \quad \mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}_\delta^h(E).$$

We will consider functions  $g, h$  such that there exist constants  $c, C$  with  $0 < c \leq \frac{g(x)}{h(x)} \leq C < \infty$  for all  $x > 0$  to be equivalent, even though they would not lead to the exact same measures. In that case we write  $g \equiv h$ .

The Hausdorff dimension of a set  $E \subseteq \mathbb{R}^n$  is the unique real number  $s$  characterized by the following properties:

- $\mathcal{H}^r(E) = +\infty$  for all  $r < s$ .
- $\mathcal{H}^t(E) = 0$  for all  $s < t$ .

Therefore, to prove that a given set  $E$  has dimension  $s$ , it is enough to check the preceding two properties, independently if  $\mathcal{H}^s(E)$  is zero, finite and positive, or infinite.

However, in general it is not true that, given a set  $E$ , there is a function  $h \in \mathbb{H}$ , such that if  $g \succ h$  then  $\mathcal{H}^g(E) = 0$ , and if  $g \prec h$ , then  $\mathcal{H}^g(E) = +\infty$ .

The difficulties arise from two results due to Besicovitch (see [Rog98] and references therein). The first says that if a set  $E$  has null  $\mathcal{H}^h$ -measure for some  $h \in \mathbb{H}$ , then there exists a function  $g$  which is dimensionally smaller

than  $h$  and for which still  $\mathcal{H}^g(E) = 0$ . Symmetrically, the second says that if a compact set  $E$  has non- $\sigma$ -finite  $\mathcal{H}^h$  measure, then there exists a function  $g \succ h$  such that  $E$  has also non- $\sigma$ -finite  $\mathcal{H}^g$  measure. These two results imply that if a set  $E$  satisfies that there exists a function  $h$  such that  $\mathcal{H}^g(E) > 0$  for any  $g \prec h$  and  $\mathcal{H}^g(E) = 0$  for any  $g \succ h$ , then it must be the case that  $0 < \mathcal{H}^h(E) < +\infty$ . This is the particular case in which  $E$  is a so called  $h$ -set (generalizing the notion of  $s$ -set for those  $s$ -dimensional sets that satisfy  $0 < H^s(E) < +\infty$ ).

But if we look for example at the set  $\mathbb{L}$  of Liouville numbers, this set is not an  $h$ -set for any  $h \in \mathbb{H}$ . These sets are called *dimensionless* (see [EK06]). In addition, it is shown in [OR06] that there are two proper nonempty subsets  $\mathbb{L}_0, \mathbb{L}_\infty \subseteq \mathbb{H}$  of dimension functions such that  $\mathcal{H}^h(\mathbb{L}) = 0$  for all  $h \in \mathbb{L}_0$  and  $\mathcal{H}^h(\mathbb{L}) = \infty$  for all  $h \in \mathbb{L}_\infty$ . This motivates the following notion of dimension partition (see [CHM10]).

## 1.2. Dimension Partition and Dimension Gaps.

**Definition 1.5.** By the *Dimension Partition* of a set  $E$  we mean a partition of  $\mathbb{H}$  into (three) sets:  $\mathcal{P}(E) = E_0 \cup E_1 \cup E_\infty$  with

- $E_0 = \{h \in \mathbb{H} : \mathcal{H}^h(E) = 0\}$ .
- $E_1 = \{h \in \mathbb{H} : 0 < \mathcal{H}^h(E) < \infty\}$ .
- $E_\infty = \{h \in \mathbb{H} : \mathcal{H}^h(E) = \infty\}$ .

Note that by the previous example, it is clear that there are sets  $E$  for which  $E_1$  is empty, reflecting the dimensionless nature of  $E$ . On the other hand,  $E_1$  is never empty for an  $h$ -set, but it is not easy to determine this partition in the general case. We also remark that it is possible to find non-comparable dimension functions  $g \not\equiv h$  and a set  $E$  with the property of being a  $g$ -set and an  $h$ -set simultaneously.

It follows that even for  $h$ -sets the dimension partition, and in particular  $E_1$ , is not completely determined. Note that the results of Besicovitch cited above imply that, for compact sets,  $E_0$  and  $E_\infty$  can be thought of as open components of the partition, and  $E_1$  as the “border” of these open components. An interesting problem is then to determine some criteria to classify the functions in  $\mathbb{H}$  into those classes.

To detect where this “border” is, we will introduce the notion of *chains* in  $\mathbb{H}$ . This notion allows to refine the notion of Hausdorff dimension by using an ordered family of dimension functions. More precisely, we have the following definition.

**Definition 1.6.** A family  $\mathcal{C} \subset \mathbb{H}$  of dimension functions will be called a *chain* if it is of the form

$$\mathcal{C} = \{h_t \in \mathbb{H} : t \in \mathbb{R}, h_s \prec h_t \iff s < t\}.$$

That is, a totally ordered one-parameter family of dimension functions.

Suppose that  $h \in \mathbb{H}$  belongs to some chain  $\mathcal{C}$  and satisfies that, for any  $g \in \mathcal{C}$ ,  $\mathcal{H}^g(E) > 0$  if  $g \prec h$  and  $\mathcal{H}^g(E) = 0$  if  $g \succ h$ . Then, even if  $h \notin E_1$ , in this chain,  $h$  does measure the size of  $E$ . It can be thought of as being “near the frontier” of both  $E_0$  and  $E_\infty$ . For example, if a set  $E$  has Hausdorff dimension  $\alpha$  but  $\mathcal{H}^\alpha(E) = 0$  or  $\mathcal{H}^\alpha(E) = \infty$ , take  $h_\alpha(x) = x^\alpha$

and  $\mathcal{C}_H = \{x^t : t \geq 0\}$ . In this chain,  $h_\alpha$  is the function that best measures the size of  $E$ .

We look for finer estimates, considering chains of dimension functions that separate sets of “the same Hausdorff dimension”. The goal will be to find very fine chains to give precise bounds on how far from an expected dimension function the Hausdorff measure drops to zero or remains positive. In this setting, the sharpness of the results are associated to the “fineness” of the chains. Consider, for example, a set  $A$  such that  $\dim_H(A) = \alpha$ , in which case it is clear that  $\mathcal{H}^{\alpha+\varepsilon}(A) = 0$  for all  $\varepsilon > 0$ . Compare the chain  $\mathcal{C}_H$  (which only detects the Hausdorff dimension of  $A$ ) to  $\mathcal{C}_{\log} = \{x^\alpha \log(\frac{1}{x})^{-\beta}, \beta \in \mathbb{R}\}$ , in which we can detect if there exists  $\beta \in \mathbb{R}$  for which  $\mathcal{H}^{x^\alpha \log^{-\beta-\varepsilon}(\frac{1}{x})}(A) = 0$  and  $\mathcal{H}^{x^\alpha \log^{-\beta+\varepsilon}(\frac{1}{x})}(A) = \infty$  for all  $\varepsilon > 0$ .

To measure this distance between dimension functions, we introduce the following notion:

**Definition 1.7.** Let  $g, h \in \mathbb{H}$  with  $g \prec h$ . Define the “gap” between  $g$  and  $h$  as

$$(3) \quad \Delta(g, h)(x) = \frac{h(x)}{g(x)}.$$

From this definition and the definition of partial order, we always have that  $\lim_{x \rightarrow 0} \Delta(g, h)(x) = 0$ , and therefore the speed of convergence to zero of  $\Delta(g, h)$  can be seen as a notion of distance between  $g$  and  $h$ .

**1.3. Generalized Furstenberg sets.** The analogous definition of Furstenberg sets in the setting of dimension functions is the following.

**Definition 1.8.** Let  $\mathfrak{h}$  be a dimension function. A set  $E \subseteq \mathbb{R}^2$  is a Furstenberg set of type  $\mathfrak{h}$ , or an  $F_{\mathfrak{h}}$ -set, if for each direction  $e \in \mathbb{S}$  there is a line segment  $\ell_e$  in the direction of  $e$  such that  $\mathcal{H}^{\mathfrak{h}}(\ell_e \cap E) > 0$ .

In [MR10] we proved that the appropriate dimension function for an  $F_{\mathfrak{h}}$  set  $E$  must be dimensionally not much smaller than  $\mathfrak{h}^2$  and  $\mathfrak{h}\sqrt{\cdot}$  (this is the generalized version of the left hand side of (1)), the latter with some additional conditions on  $\mathfrak{h}$ . There we also presented precise bounds on the dimensional gaps. These results are valid for all dimension functions  $\mathfrak{h}$  - minding the small restrictions on the decay of  $\mathfrak{h}$  for the second bound - regardless of the fact that  $\mathfrak{h}$  is (or is not) zero-dimensional. We also exhibited an example of a zero dimensional Furstenberg-type set with 2 points in each direction, proving that for this case,  $\inf \dim E = 0$  contradicting the naïve limiting argument in equation (1), that  $\gamma(0) = \frac{1}{2}$ . (It seems feasible to find an argument to also construct a zero dimensional  $F^K$ -set for  $3 \leq K \leq \aleph_0$  points as remarked by Keleti and Máthé [Shm11]) However, if we consider zero-dimensional Furstenberg-type sets belonging to  $F_{\mathfrak{h}_\gamma}$ , where  $\mathfrak{h}_\gamma \in \mathbb{H}_0$  is defined by  $\mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma(\frac{1}{x})}$ , we showed that any  $E \in F_{\mathfrak{h}_\gamma}$  must have  $\dim_H(E) \geq \frac{1}{2}$ .

In the present work we look at a refinement of the upper bound for the dimension of Furstenberg sets. Since we are looking for upper bounds on a class of Furstenberg sets, the aim will be to explicitly construct a very small set belonging to the given class.

We first consider the classical case of power functions,  $x^\alpha$ , for  $\alpha > 0$ . Recall that for this case, the known upper bound implies that, for any positive  $\alpha$ , there is a set  $E \in F_\alpha$  such that  $\mathcal{H}^{\frac{1+3\alpha}{2}+\varepsilon}(E) = 0$  for any  $\varepsilon > 0$ . We improve this estimate in terms of logarithmic gaps by showing that there is no need to take a “power like” step from the critical dimension. For any  $\alpha > 0$  consider  $h_\theta(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}(\frac{1}{x})$ . Then, there is an  $F_\alpha$ -set  $E$  such that  $\mathcal{H}^{h_\theta}(E) = 0$  for any  $\theta > \frac{1+3\alpha}{2}$  (Theorem 2.4). Note that  $x^{\frac{1+3\alpha}{2}} \prec h_\theta \prec x^{\frac{1+3\alpha}{2}+\varepsilon}$  for all  $\varepsilon > 0$ .

We will also focus at the endpoint  $\alpha = 0$ , and give a complete answer about the size of a class of Furstenberg sets by proving that, for any given  $\gamma > 0$ , there exists a set  $E_\gamma \subseteq \mathbb{R}^2$  such that (Theorem 3.2)

$$(4) \quad E_\gamma \in F_{\mathfrak{h}_\gamma} \text{ for } \mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma(\frac{1}{x})} \text{ and } \dim_H(E_\gamma) \leq \frac{1}{2}.$$

Note that, when  $\gamma \rightarrow +\infty$ , the dimension functions  $\mathfrak{h}_\gamma; \gamma > 0$  increase, “approaching” from below the positive-dimensional case. However, when  $\gamma \rightarrow 0$ , the functions decrease, “approaching” the sets having finitely many (or countably many) points in each direction. In this latter case, a zero-dimensional Furstenberg set can be constructed [Shm11]. We will provide an example of an even smaller Furstenberg set  $E_{\mathfrak{h}}$  (which naturally depends on the choice of  $\mathfrak{h} \in \mathbb{H}_0$ ) that has 2 points in each direction, but such that  $\mathcal{H}^{\mathfrak{h}}(E_{\mathfrak{h}}) = 0$  (see Example 3.4).

As usual, we will use the notation  $A \lesssim B$  to indicate that there is a constant  $C > 0$  such that  $A \leq CB$ , where the constant is independent of  $A$  and  $B$ . By  $A \sim B$  we mean that both  $A \lesssim B$  and  $B \lesssim A$  hold.

The paper is organized as follows. In Section 2, for a given  $\alpha > 0$ , we develop the main construction of small Furstenberg  $\alpha$  sets, and obtain dimension estimates for the class  $F_\alpha$ ,  $\alpha > 0$ . In Section 3 we first show that we can refine the argument of the previous section to include the zero-dimensional functions  $\mathfrak{h}_\gamma$  defined in (4). We then continue to improve on the construction given in [MR10] to construct extremely small Furstenberg sets that contain 2 points in each direction and we look at the same problem from the point of view of Packing dimension and compare the very different behaviour. Section 4 is devoted to the proof of an important lemma regarding the size of zero dimensional fibers.

## 2. UPPER BOUNDS FOR FURSTENBERG-TYPE SETS

In this section, given  $\alpha > 0$  we will focus our attention to the upper dimension estimate for  $F_\alpha$  sets. As stated in (1) it was shown that if  $\gamma(\alpha)$  is the infimum of all the possible values for the Hausdorff dimension of  $F_\alpha$ -sets, then

$$(5) \quad \max\{2\alpha; \frac{1}{2} + \alpha\} \leq \gamma(\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \quad \alpha \in (0, 1].$$

We will concentrate on the right hand side of this inequality, which has been proved by showing that there exists a set  $E$  in  $F_\alpha$ , such that  $\mathcal{H}^s(E) = 0$  for any  $s > \frac{1+3\alpha}{2}$ .

However, by the result of Besicovitch cited in the Introduction, we know that it is not going to be true that  $\mathcal{H}^h(E) = 0$  for any  $h \succ x^{\frac{1+3\alpha}{2}}$ .

By refining the arguments of Wolff in Theorem 2.4, if  $\theta > \frac{1+3\alpha}{2}$  we are able to exhibit a set  $E$  in  $F_\alpha$ , such that for  $h_\theta := x^{\frac{1+3\alpha}{2}} \log^{-\theta}(\frac{1}{x})$  we have that  $\mathcal{H}^{h_\theta}(E) = 0$ . Note that  $h_\theta \prec x^s$  for any  $s > \frac{1+3\alpha}{2}$ .

We begin with a preliminary lemma about a very well distributed (mod 1) sequence.

**Lemma 2.1.** *For  $n \in \mathbb{N}$  and any real number  $x \in [0, 1]$ , there is a pair  $0 \leq j, k \leq n-1$  such that*

$$\left| x - \left( \sqrt{2} \frac{k}{n} - \frac{j}{n} \right) \right| \leq \frac{\log(n)}{n^2}.$$

This lemma is a consequence of Theorem 3.4 of [KN74], p125, in which an estimate is given about the discrepancy of the fractional part of the sequence  $\{n\alpha\}_{n \in \mathbb{N}}$  where  $\alpha$  is a irrational of a certain type.

We also need to introduce the notion of  $G$ -sets, a common ingredient in the construction of Kakeya and Furstenberg sets.

**Definition 2.2.** A  $G$ -set is a compact set  $E \subseteq \mathbb{R}^2$  which is contained in the strip  $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$  such that for any  $m \in [0, 1]$  there is a line segment contained in  $E$  connecting  $x = 0$  with  $x = 1$  of slope  $m$ , i.e.

$$\forall m \in [0, 1] \exists b \in \mathbb{R} : mx + b \in E, \forall x \in [0, 1].$$

Finally we need some notation for a thickened line.

**Definition 2.3.** Given a line segment  $\ell(x) = mx + b$ , we define the  $\delta$ -tube associated to  $\ell$  as

$$S_\ell^\delta := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1; |y - (mx + b)| \leq \delta\}.$$

Now we are ready to prove the main result of this section.

**Theorem 2.4.** *For  $\alpha \in (0, 1]$  and  $\theta > 0$ , define  $h_\theta(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}(\frac{1}{x})$ . Then, if  $\theta > \frac{1+3\alpha}{2}$ , there exists a set  $E \in F_\alpha$  with  $\mathcal{H}^{h_\theta}(E) = 0$ .*

*Proof.* Fix  $n \in \mathbb{N}$  and let  $n_j$  be a sequence such that  $n_{j+1} > n_j^j$ . We consider  $T$  to be the set defined as follows:

$$T = \left\{ x \in \left[ \frac{1}{4}, \frac{3}{4} \right] : \forall j \exists p, q; q \leq n_j^\alpha; \left| x - \frac{p}{q} \right| < \frac{1}{n_j^2} \right\}.$$

It can be seen that  $\dim_H(T) = \alpha$  (see Section 4, Theorem 4.8).

If  $\varphi(t) = \frac{1-t}{t\sqrt{2}}$  and  $D = \varphi^{-1}([\frac{1}{4}, \frac{3}{4}])$ , we have that  $\varphi : D \rightarrow [\frac{1}{4}, \frac{3}{4}]$  is bi-Lipschitz. Therefore the set

$$T' = \left\{ t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T \right\} = \varphi^{-1}(T)$$

also has Hausdorff dimension  $\alpha$ .

The main idea of our proof, is to construct a set for which we have, essentially, a copy of  $T'$  in each direction and simultaneously keep some optimal covering property.

Define, for each  $n \in \mathbb{N}$ ,

$$\Gamma_n := \left\{ \frac{p}{q} \in \left[ \frac{1}{4}, \frac{3}{4} \right], q \leq n^\alpha \right\}$$

and

$$Q_n = \left\{ t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n \right\} = \varphi^{-1}(\Gamma_n).$$

To count the elements of  $\Gamma_n$  (and  $Q_n$ ), we take into account that

$$\sum_{j=1}^{\lfloor n^\alpha \rfloor} j \leq \frac{1}{2} \lfloor n^\alpha \rfloor (\lfloor n^\alpha \rfloor + 1) \lesssim \lfloor n^\alpha \rfloor^2 \leq n^{2\alpha}.$$

Therefore,  $\#(Q_n) \lesssim n^{2\alpha}$ .

For  $0 \leq j, k \leq n-1$ , define the line segments

$$\ell_{jk}(x) := (1-x)\frac{j}{n} + x\sqrt{2}\frac{k}{n} \text{ for } x \in [0, 1],$$

and their  $\delta_n$ -tubes  $S_{\ell_{jk}}^{\delta_n}$  with  $\delta_n = \frac{\log(n)}{n^2}$ . We will use during the proof the notation  $S_{jk}^n$  instead of  $S_{\ell_{jk}}^{\delta_n}$ . Also define

$$(6) \quad G_n := \bigcup_{jk} S_{jk}^n.$$

Note that, by Lemma 2.1, all the  $G_n$  are  $G$ -sets.

For each  $t \in Q_n$ , we look at the points  $\ell_{jk}(t)$ , and define the set  $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$ . Clearly,  $\#(S(t)) \leq n^2$ . But if we note that, if  $t \in Q_n$ , then

$$0 \leq \frac{\ell_{jk}(t)}{t\sqrt{2}} = \frac{1-t}{t\sqrt{2}} \frac{j}{n} + \frac{k}{n} = \frac{p}{q} \frac{j}{n} + \frac{k}{n} = \frac{pj + kq}{nq} < 2,$$

we can bound  $\#(S(t))$  by the number of non-negative rationals smaller than 2 of denominator  $qn$ . Since  $q \leq n^\alpha$ , we have  $\#(S(t)) \leq n^{1+\alpha}$ . Considering all the elements of  $Q_n$ , we obtain  $\# \left( \bigcup_{t \in Q_n} S(t) \right) \lesssim n^{1+3\alpha}$ . Let us define

$$\Lambda_n := \left\{ (x, y) \in G_n : |x - t| \leq \frac{C}{n^2} \text{ for some } t \in Q_n \right\}.$$

**Claim 2.5.** *For each  $n$ , take  $\delta_n = \frac{\log(n)}{n^2}$ . Then  $\Lambda_n$  can be covered by  $L_n$  balls of radio  $\delta_n$  with  $L_n \lesssim n^{1+3\alpha}$ .*

To see this, it suffices to set a parallelogram on each point of  $S(t)$  for each  $t$  in  $Q_n$ . The lengths of the sides of the parallelogram are of order  $n^{-2}$  and  $\frac{\log(n)}{n^2}$ , so their diameter is bounded by a constant times  $\frac{\log(n)}{n^2}$ , which proves the claim.

We can now begin with the recursive construction that leads to the desired set. Let  $F_0$  be a  $G$ -set written as

$$F_0 = \bigcup_{i=1}^{M_0} S_{\ell_i^0}^{\delta^0},$$

(the union of  $M_0$   $\delta^0$ -thickened line segments  $\ell_i^0 = m_i^0 + b_i^0$  with appropriate orientation). Each  $F_j$  to be constructed will be a  $G$ -set of the form

$$F_j := \bigcup_{i=1}^{M_j} S_{\ell_i^j}^{\delta^j}, \quad \text{with } \ell_i^j = m_i^j + b_i^j.$$

Having constructed  $F_j$ , consider the  $M_j$  affine mappings

$$A_i^j : [0, 1] \times [-1, 1] \rightarrow S_{\ell_i^j}^{\delta_j} \quad 1 \leq i \leq M_j,$$

defined by

$$A_i^j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m_i^j & \delta_j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ b_i^j \end{pmatrix}.$$

We choose  $n_{j+1}$  large enough to satisfy

$$(7) \quad \log \log(n_{j+1}) > M_j$$

and apply  $A_i^j$  to the sets  $G_{n_{j+1}}$  defined in (6) to obtain

$$F_{j+1} = \bigcup_{i=1}^{M_j} A_i^j(G_{n_{j+1}}).$$

Since  $G_{n_{j+1}}$  is a union of thickened line segments, we have that

$$F_{j+1} = \bigcup_{i=1}^{M_{j+1}} S_{\ell_i^{j+1}}^{\delta_{j+1}},$$

for an appropriate choice of  $M_{j+1}$ ,  $\delta_{j+1}$  and  $M_{j+1}$  line segments  $\ell_i^{j+1}$ . From the definition of the mappings  $A_i^j$  and since the set  $G_{n_{j+1}}$  is a  $G$ -set, we conclude that  $F_{j+1}$  is also a  $G$ -set. Define

$$E_j := \{(x, y) \in F_j : x \in T'\}.$$

To cover  $E_j$ , we note that if  $(x, y) \in E_j$ , then  $x \in T'$ , and therefore there exists a rational  $\frac{p}{q} \in \Gamma_{n_j}$  with

$$\frac{1}{n_j^2} > \left| \frac{1-x}{x\sqrt{2}} - \frac{p}{q} \right| = |\varphi(x) - \varphi(r)| \geq \frac{|x-r|}{\sqrt{2}}, \quad \text{for some } r \in Q_{n_j}.$$

Therefore  $(x, y) \in \bigcup_{i=1}^{M_{j-1}} A_i^{j-1}(\Lambda_{n_j})$ , so we conclude that  $E_j$  can be covered by  $M_{j-1}n_j^{1+3\alpha}$  balls of diameter at most  $\frac{\log(n_j)}{n_j^2}$ . Since we chose  $n_j$  such that  $\log \log(n_j) > M_{j-1}$ , we obtain that  $E_j$  admits a covering by  $\log \log(n_j)n_j^{1+3\alpha}$  balls of the same diameter. Therefore, if we set  $F = \bigcap_j F_j$  and  $E := \{(x, y) \in F : x \in T'\}$  we obtain that

$$\begin{aligned} \mathcal{H}_{\delta_j}^{h_\theta}(E) &\lesssim n_j^{1+3\alpha} \log \log(n_j) h_\theta \left( \frac{\log(n_j)}{n_j^2} \right) \\ &\lesssim n_j^{1+3\alpha} \log \log(n_j) \left( \frac{\log(n_j)}{n_j^2} \right)^{\frac{1+3\alpha}{2}} \log^{-\theta} \left( \frac{n_j^2}{\log(n_j)} \right) \\ &\lesssim \log \log(n_j) \log(n_j)^{\frac{1+3\alpha}{2}-\theta} \lesssim \log^{\frac{1+3\alpha}{2}+\varepsilon-\theta}(n_j) \end{aligned}$$

for  $x \geq K = K(\varepsilon)$ . Therefore, for any  $\theta > \frac{1+3\alpha}{2}$ , the last expression goes to zero. In addition,  $F$  is a  $G$ -set, so it must contain a line segment in each direction  $m \in [0, 1]$ . If  $\ell$  is such a line segment, then

$$\dim_H(\ell \cap E) = \dim_H(T') \geq \alpha.$$



The final set of the proposition is obtained by taking eight copies of  $E$ , rotated to achieve *all* the directions in  $\mathbb{S}$ .  $\square$

### 3. UPPER BOUNDS FOR SMALL FURSTENBERG-TYPE SETS

In this section we will focus on the class  $F_\alpha$  at the endpoint  $\alpha = 0$ . Note that all preceding results involved only the case for which  $\alpha > 0$ . Introducing the generalized Hausdorff measures, we are able to handle an important class of Furstenberg type sets in  $F_0$ .

We will look first at the 0-dimensional case, but in which we request the set to contain uncountably many points. For this, it will be necessary to put in each direction some set with many points but with certain structure.

The proof relies on Theorem 2.4, but we must replace the set  $T$  by a generalized version of it. More precisely, we will need the following lemma.

**Lemma 3.1.** *Let  $r > 1$  and consider the sequence  $\mathbf{n} = \{n_j\}$  defined by  $n_j = e^{\frac{1}{2}n_{j-1}^{\frac{4}{r}}}$ , the function  $\mathfrak{f}(x) = \log(x^2)^{\frac{r}{2}}$  and the set*

$$T = \left\{ x \in \left[ \frac{1}{4}, \frac{3}{4} \right] \setminus \mathbb{Q} : \forall j \exists p, q ; q \leq \mathfrak{f}(n_j); |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\}.$$

*Then we have that  $\mathcal{H}^{\mathfrak{h}}(T) > 0$  for  $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$ .*

We postpone the proof of this lemma to the next section. With this lemma, we are able to prove the main result of this section. We have the next theorem.

**Theorem 3.2.** *Let  $\mathfrak{h} = \frac{1}{\log(\frac{1}{x})}$ . There exists a set  $E \in F_{\mathfrak{h}}$  such that  $\dim_H(E) \leq \frac{1}{2}$ .*

*Proof.* We will use essentially a copy of  $T$  in each direction in the construction of the desired set to fulfill the conditions required to be an  $F_{\mathfrak{h}}$ -set. Let  $T$  be the set defined in Lemma 3.1. Define  $T'$  as

$$T' = \left\{ t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T \right\} = \varphi^{-1}(T),$$

where  $\varphi$  is the same bi-Lipschitz function from the proof of Theorem 2.4. Then  $T'$  has positive  $\mathcal{H}^{\mathfrak{h}}$ -measure. Let us define the corresponding sets of Theorem 2.4 for this generalized case.

$$\Gamma_n := \left\{ \frac{p}{q} \in \left[ \frac{1}{4}, \frac{3}{4} \right], q \leq \mathfrak{f}(n) \right\},$$

$$Q_n = \left\{ t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n \right\} = \varphi^{-1}(\Gamma_n).$$

Now the estimate is  $\#(Q_n) \lesssim \mathfrak{f}^2(n) = \log^r(n^2) \sim \log^r(n)$ , since

$$\sum_{j=1}^{\lfloor \mathfrak{f}(n) \rfloor} j \leq \frac{1}{2} \lfloor \mathfrak{f}(n) \rfloor (\lfloor \mathfrak{f}(n) \rfloor + 1) \lesssim \lfloor \mathfrak{f}(n) \rfloor^2.$$

For each  $t \in Q_n$ , define  $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$ . If  $t \in Q_n$ , following the previous ideas, we obtain that

$$\#(S(t)) \lesssim n \log^{\frac{r}{2}}(n),$$

and therefore

$$\# \left( \bigcup_{t \in Q_n} S(t) \right) \lesssim n \log(n)^{\frac{3r}{2}}.$$

Now we estimate the size of a covering of

$$\Lambda_n := \left\{ (x, y) \in G_n : |x - t| \leq \frac{C}{n^2} \text{ for some } t \in Q_n \right\}.$$

For each  $n$ , take  $\delta_n = \frac{\log(n)}{n^2}$ . As before, the set  $\Lambda_n$  can be covered with  $L_n$  balls of radio  $\delta_n$  with  $L_n \lesssim n \log(n)^{\frac{3r}{2}}$ .

Once again, define  $F_j$ ,  $F$ ,  $E_j$  and  $E$  as before. Now the sets  $F_j$  can be covered by less than  $M_{j-1} n_j \log(n_j)^{\frac{3r}{2}}$  balls of diameter at most  $\frac{\log(n_j)}{n_j^2}$ . Now we can verify that, since each  $G_n$  consist of  $n^2$  tubes, we have that  $M_j = M_0 n_1^2 \cdots n_j^2$ . We can also verify that the sequence  $\{n_j\}$  satisfies the relation  $\log n_{j+1} \geq M_j = M_0 n_1^2 \cdots n_j^2$ , and therefore we have the bound

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \lim_j \frac{\log \left( \log(n_j) n_j \log(n_j)^{\frac{3r}{2}} \right)}{\log \left( n_j^2 \log^{-1}(n_j) \right)} = \frac{1}{2},$$

where  $\underline{\dim}_B$  stands for the lower box dimension. Finally, for any  $m \in [0, 1]$  we have a line segment  $\ell$  with slope  $m$  contained in  $F$ . It follows that  $\mathcal{H}^b(\ell \cap E) = \mathcal{H}^b(T') > 0$ .  $\square$

We remark that the argument in this particular result is essentially the same needed to obtain the family of Furstenberg sets  $E_\gamma \in F_{\mathfrak{h}_\gamma}$  for  $\mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma(\frac{1}{x})}$ ,  $\gamma \in \mathbb{R}_+$ , such that  $\dim_H(E_\gamma) \leq \frac{1}{2}$  announced in the introduction.

We will now look at the case of finite number of points in each direction.

**3.1. The case  $\alpha = 0$ ,  $K$  points.** Let us begin with the definition of the class  $F^K$ .

**Definition 3.3.** For  $K \in \mathbb{N}$ ,  $K \geq 2$ , a set will be a  $F^K$ -set or a Furstenberg set of type  $K$  if for any direction  $e \in \mathbb{S}$ , there are at least  $K$  points contained in  $E$  lined up in the direction of  $e$ .

This case contrasts with the previous case, since we show that in this case  $\frac{1}{2}$  is not longer the smallest possible dimension. Already in [MR10] we proved that there is a  $F^2$ -set with zero Hausdorff dimension. We will generalize this example to obtain even smaller  $F^2$  sets. Namely, for any  $h \in \mathbb{H}_0$ , there exists  $G$  in  $F^2$  such that  $\mathcal{H}^h(G) = 0$ . It is clear that the set  $G$  will depend on the choice of  $h$ .

**Example 3.4.** Given a function  $h \in \mathbb{H}$ , we will construct two small sets  $E, F \subseteq [0, 1]$  with  $\mathcal{H}^h(E) = \mathcal{H}^h(F) = 0$  and such that  $[0, 1] \subseteq E + F$ . Consider now  $G = E \times \{1\} \cup -F \times \{0\}$ . Clearly, we have that  $\mathcal{H}^h(G) = 0$ , and contains two points in every direction  $\theta \in [0; \frac{\pi}{4}]$ . For, if  $\theta \in [0; \frac{\pi}{4}]$ , let  $c = \tan(\theta)$ , so  $c \in [0, 1]$ . By the choice of  $E$  and  $F$ , we can find  $x \in E$  and  $y \in F$  with  $c = x + y$ . The points  $(-y, 0)$  and  $(x, 1)$  belong to  $G$  and determine a segment in the direction  $\theta$ .

We define

$$E := \{x \in [0, 1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ even}\}$$

and

$$F := \{x \in [0, 1] : r_j = 0 \text{ if } m_k + 1 \leq j \leq m_{k+1}; k \text{ odd}\}.$$

Here  $\{m_k; m_0 = 0\}_k$  is an increasing sequence such that  $m_k \rightarrow +\infty$ . Now we estimate the size of the set  $E$ . Given  $k \in \mathbb{N}$ ,  $k$  even, define  $\ell_k = m_k - m_{k-1} + \dots + m_2 - m_1$ . It is clear that  $E$  can be covered by  $2^{\ell_k}$  intervals of length  $2^{-m_{k+1}}$ . Therefore, if the sequence  $m_k$  increases fast enough, then

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \lim_k \frac{\log(2^{\ell_k})}{\log(2^{m_{k+1}})} \lesssim \lim_k \frac{2^{\ell_k}}{2^{m_{k+1}}} = 0.$$

Since the same argument shows that  $\dim_H(F) = 0$ , this estimate proves that the set  $G$  has Hausdorff dimension equal to zero. Now, for the finer estimate on the  $\mathcal{H}^h$ -measure of the set, we must impose a more restrictive condition on the sequence  $\{m_k\}$ .

Recall that the covering property implies that, for a given  $h \in \mathbb{H}$ , we have that

$$\mathcal{H}^h(E) \leq 2^{\ell_k} h(2^{-m_{k+1}}).$$

Therefore we need to choose a sequence  $\{m_j\}$ , depending on  $h$ , such that the above quantity goes to zero with  $k$ . Since  $\ell_k \leq m_k$ , we can define recursively the sequence  $\{m_k\}$  to satisfy the relation

$$2^{m_k} h(2^{-m_{k+1}}) = \frac{1}{k}.$$

This last condition is equivalent to  $m_{k+1} = \log \left( \frac{1}{h^{-1}(\frac{1}{k2^{m_k}})} \right)$ . As an concrete example, take  $h(x) = \frac{1}{\log(\frac{1}{x})}$ . In this case we obtain that the sequence  $\{m_k\}$  can be defined as

$$m_{k+1} = k2^{m_k}.$$

**3.2. Remark about the Packing dimension for small sets.** It is worthy to note here that if we were to measure the size of Furstenberg sets with the packing dimension, the situation is absolutely different. More precisely, for  $K \geq 2$ , any  $F^K$ -set  $E \subset \mathbb{R}^2$  must have  $\dim_P(E) \geq \frac{1}{2}$ . For, if  $E$  is an  $F^2$  set, then the map  $\varphi$  defined by  $\varphi(a, b) = \frac{a-b}{\|a-b\|}$  is Lipschitz when restricted to  $G_\varepsilon := E \times E \setminus \{(x, y) \in E \times E : \|(x, y) - (a, a)\| < \varepsilon; a \in E\}$ . Roughly, we are considering the map that recovers the set of directions but restricted “off the diagonal”. It is clear that we can assume without loss of generality that all the pairs are the endpoints of unit line segments. Therefore, since  $E$  is an  $F^K$ -set,  $\varphi(G_\varepsilon) = \mathbb{S}$  if  $\varepsilon$  is small enough. We obtain the inequality

$$1 = \dim_H(\mathbb{S}) \leq \dim_H(G_\varepsilon) \leq \dim_H(E \times E).$$

The key point is the product formulae for Hausdorff and Packing dimensions. We obtain that

$$(8) \quad 1 \leq \dim_H(E \times E) \leq \dim_H(E) + \dim_P(E) \leq 2 \dim_P(E)$$

and then  $\dim_P(E) \geq \frac{1}{2}$ . It also follows that if we achieve small Hausdorff dimension then the Packing dimension is forced to increase. In particular,

the  $F^2$ -set constructed in [MR10] has Hausdorff dimension 0 and therefore it has Packing dimension 1.

By means of an example we illustrate the relation between Hausdorff and Packing dimension for Furstenberg sets. It can be understood as optimal in the sense of obtaining the smallest possible dimensions, both Hausdorff and Packing. Recall that from (8) we have that for any  $K \geq 2$ , any Furstenberg set  $E$  of the class  $F^K$  must satisfy

$$1 \leq \dim_H(E) + \dim_P(E) \leq 2 \dim_P(E).$$

In fact, there is an  $F^2$  set  $E$  such that  $\dim_H(E) = \frac{1}{2} = \dim_P(E)$ .

**Example 3.5.** The construction is essentially the same as in Example 3.4, but we use two different sets to obtain all directions. Let  $A$  be the set of all the numbers whose expansion in base 4 uses only the digits 0 and 1. On the other hand, let  $B$  the set of those numbers which only uses the digits 0 and 2. Both sets have Packing and Hausdorff dimension equal to  $\frac{1}{2}$  and  $[0, 1] \subseteq A + B$ . The construction follows then the same pattern as in the previous example.

#### 4. PROOF OF LEMMA 3.1

The purpose of this section is to prove Lemma 3.1. Our proof relies on a variation of a Jarník type theorem on Diophantine approximation. We begin with some preliminary results on Cantor type constructions that will be needed.

**4.1. Cantor sets.** In this section we introduce the construction of sets of Cantor type in the spirit of [Fal03]. By studying two quantities, the number of children of a typical interval and some separation property, we obtain sufficient conditions on these quantities that imply the positivity of the  $h$ -dimensional measure for a test function  $h \in \mathbb{H}$ .

We will need a preliminary elemental lemma about concave functions. The proof is straightforward.

**Lemma 4.1.** *Let  $h \in \mathbb{H}$  be a concave dimension function. Then*

$$\min\{a, b\} \leq \frac{a}{h(a)} h(b) \quad \text{for any } a, b \in \mathbb{R}_+.$$

*Proof.* We consider two separate cases:

- If  $b \geq a$  then  $\frac{a}{h(a)} h(b) \geq \frac{a}{h(a)} h(a) = a = \min\{a, b\}$ .
- If  $a > b$ , then  $\frac{a}{h(a)} h(b) \geq \frac{b}{h(b)} h(b) = b = \min\{a, b\}$  by concavity of  $h$ .

□

The following lemma is a natural extension of the “Mass Distribution Principle” to the dimension function setting.

**Lemma 4.2** ( $h$ -dimensional mass distribution principle). *Let  $E \subseteq \mathbb{R}^n$  be a set,  $h \in \mathbb{H}$  and  $\mu$  a probability measure on  $E$ . Let  $\varepsilon > 0$  and  $c > 0$  be positive constants such that for any  $U \subseteq \mathbb{R}^n$  with  $\text{diam}(U) < \varepsilon$  we have*

$$\mu(U) \leq ch(\text{diam}(U)).$$

*Then  $\mathcal{H}^h(E) > 0$ .*

*Proof.* For any  $\delta$ -covering we have

$$0 < \mu(E) \leq \sum_i \mu(U) \leq c \sum_i h(\text{diam}(U)).$$

Then  $\mathcal{H}_\delta^h > \frac{\mu(E)}{c}$  and therefore  $\mathcal{H}^h(E) > 0$ .  $\square$

Now we present the construction of a Cantor-type set (see Example 4.6 in [Fal03]).

**Lemma 4.3.** *Let  $\{E_k\}$  be a decreasing sequence of closed subsets of the unit interval. Set  $E_0 = [0, 1]$  and suppose that the following conditions are satisfied:*

- (1) *Each  $E_k$  is a finite union of closed intervals  $I_j^k$ .*
- (2) *Each level  $k - 1$  interval contains at least  $m_k$  intervals of level  $k$ . We will refer to this as the “children” of an interval.*
- (3) *The gaps between the intervals of level  $k$  are at least of size  $\varepsilon_k$ , with  $0 < \varepsilon_{k+1} < \varepsilon_k$ .*

Let  $E = \bigcap_k E_k$ . Define, for a **concave** dimension function  $h \in \mathbb{H}$ , the quantity

$$D_k^h := m_1 \cdot m_2 \cdots m_{k-1} h(\varepsilon_k m_k).$$

If  $\liminf_k D_k^h > 0$ , then  $\mathcal{H}^h(E) > 0$ .

*Proof.* The idea is to use the version of the mass distribution principle from Lemma 4.2. Clearly we can assume that the property (2) of Lemma 4.3 holds for *exactly*  $m_k$  intervals. So we can define a mass distribution on  $E$  assigning a mass of  $\frac{1}{m_1 \cdots m_k}$  to each of the  $m_1 \cdots m_k$  intervals of level  $k$ . Now, for any interval  $U$  with  $0 < |U| < \varepsilon_1$ , take  $k$  such  $\varepsilon_k < |U| < \varepsilon_{k-1}$ . We will estimate the number of intervals of level  $k$  that could have non-empty intersection with  $U$ . For that, we note the following:

- $U$  intersects at most one  $I_j^{k-1}$ , since  $|U| < \varepsilon_{k-1}$ . Therefore it can intersect at most  $m_k$  children of  $I_j^{k-1}$ .
- Suppose now that  $U$  intersects  $L$  intervals of level  $k$ . Then it must contain  $(L - 1)$  gaps of size at least  $\varepsilon_k$ . Therefore,  $L - 1 \leq \frac{|U|}{\varepsilon_k}$ . Consequently  $|U|$  intersects at most  $\frac{|U|}{\varepsilon_k} + 1 \leq 2\frac{|U|}{\varepsilon_k}$  intervals of level  $k$ .

From these two observations, we conclude that

$$\mu(U) \leq \frac{1}{m_1 \cdots m_k} \min \left\{ m_k, \frac{2|U|}{\varepsilon_k} \right\} = \frac{1}{m_1 \cdots m_k \varepsilon_k} \min \{ \varepsilon_k m_k, 2|U| \}.$$

Now, by the concavity of  $h$ , we obtain

$$\min \{ \varepsilon_k m_k, 2|U| \} \leq \frac{\varepsilon_k m_k}{h(\varepsilon_k m_k)} h(2|U|).$$

In addition (also by concavity),  $h$  is doubling, so  $h(2|U|) \lesssim h(|U|)$  and then

$$\mu(U) \lesssim \frac{\varepsilon_k m_k h(|U|)}{m_1 \cdots m_k \varepsilon_k h(\varepsilon_k m_k)} = \frac{h(|U|)}{m_1 \cdots m_{k-1} h(\varepsilon_k m_k)} = \frac{h(|U|)}{D_k^h}.$$

Finally, if  $\liminf_k D_k^h > 0$ , there exists  $k_0$  such  $\frac{1}{D_k^h} \leq C$  for  $k \geq k_0$  and we can use the mass distribution principle with  $C$  and  $\varepsilon = \varepsilon_{k_0}$ .  $\square$

**Remark 4.4.** In the particular case of  $h(x) = x^s$ ,  $s \in (0, 1)$  we recover the result of [Fal03], where the parameter  $s$  can be expressed in terms of the sequences  $m_k$  and  $\varepsilon_k$ . For the set constructed in Lemma 4.3, we have

$$(9) \quad \dim_H(E) \geq \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}.$$

**4.2. Diophantine approximation - Jarník's Theorem.** The central problem in the theory of Diophantine approximation is, at its simplest level, to approximate irrational numbers by rationals. A classical theorem due to Jarník in this area is the following (see [Fal86]), which provides a result on the size of the set of real numbers that are well approximable:

**Theorem 4.5.** *For  $\beta \geq 2$ , define the following set:*

$$B_\beta = \left\{ x \in [0, 1] \setminus \mathbb{Q} : \|xq\| < \frac{1}{q^{\beta-1}} \text{ for infinitely many } q \in \mathbb{Z} \right\}.$$

*Then  $\dim_H(B_\beta) = \frac{2}{\beta}$ .*

In fact, there are several results (see [Khi24], [Jar31]) regarding the more general problem of estimating the dimension partition for the set

$$(10) \quad B_{\mathfrak{g}} := \left\{ x \in [0, 1] \setminus \mathbb{Q} : \|xq\| \leq \frac{q}{\mathfrak{g}(q)} \text{ for infinitely many } q \in \mathbb{N} \right\},$$

where  $\mathfrak{g}$  is any positive increasing function.

We can therefore see Lemma 3.1 as a result on the size of a set of well approximable numbers. We will derive the proof from the following proposition, where we prove a lower bound estimate for the set  $B_{\mathfrak{g}}$ . Precisely, for  $\mathfrak{h}(x) = \frac{1}{(\mathfrak{g}^{-1}(\frac{1}{x}))^2}$  we will find conditions on  $\Delta(h, \mathfrak{h}) = \frac{\mathfrak{h}}{h}$  with  $h \prec \mathfrak{h}$  to ensure that  $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$ .

**Proposition 4.6.** *Let  $\mathfrak{g}$  be a positive, increasing function satisfying*

$$(11) \quad \mathfrak{g}(x) \gg x^2 \quad (x \gg 1)$$

*and*

$$(12) \quad \mathfrak{g}^{-1}(ab) \lesssim \mathfrak{g}^{-1}(a) + \mathfrak{g}^{-1}(b) \text{ for all } a, b \geq 1.$$

*Define  $B_{\mathfrak{g}}$  as in (10) and let  $h \in \mathbb{H}_d$  such that  $h \prec \mathfrak{h}(x) = \frac{1}{(\mathfrak{g}^{-1}(\frac{1}{x}))^2}$ .*

*Consider a sequence  $\{n_k\}$  that satisfies:*

- (A)  $n_k \geq 3\mathfrak{g}(2n_{k-1})$ .
- (B)  $\log(n_k) \leq \mathfrak{g}(n_{k-1})$ .

*If  $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)} = \frac{1}{h(x)\mathfrak{g}^{-1}(\frac{1}{x})^2}$  satisfies*

$$(13) \quad \lim_k \frac{1}{6^k \mathfrak{g}^2(n_{k-2}) \Delta(h, \mathfrak{h}) \left( \frac{1}{\log(n_k) \mathfrak{g}(n_{k-1})} \right)} > 0,$$

*then  $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$ .*

*Proof.* Define

$$(14) \quad G_q := \{x \in [0, 1] \setminus \mathbb{Q} : \|xq\| \leq \frac{q}{\mathfrak{g}(q)}\}.$$

For each  $q \in \mathbb{N}$ ,  $G_q$  is the union of  $q - 1$  intervals of length  $2\mathfrak{g}(q)^{-1}$  and two more intervals of length  $\mathfrak{g}(q)^{-1}$  at the endpoints of  $[0, 1]$ :

$$G_q = \left(0, \frac{1}{\mathfrak{g}(q)}\right) \cup \left(\frac{1}{\mathfrak{g}(q)}, 1\right) \cup \bigcup_{1 \leq r \leq q-1} \left(\frac{r}{q} - \frac{1}{\mathfrak{g}(q)}, \frac{r}{q} + \frac{1}{\mathfrak{g}(q)}\right)$$

Now define for each  $q$

$$G'_q := G_q \cap \left(\frac{1}{\mathfrak{g}(q)}, 1 - \frac{1}{\mathfrak{g}(q)}\right).$$

Now, for each  $n \in \mathbb{N}$  consider two prime numbers  $p_1, p_2$  such that  $n \leq p_1 < p_2 < 2n$ . We will prove that  $G'_{p_1}$  and  $G'_{p_2}$  are disjoint and well separated. Note that if  $\frac{r_1}{p_1}$  and  $\frac{r_2}{p_2}$  are centers of two of the intervals belonging to  $G'_{p_1}$  and  $G'_{p_2}$ , we have

$$\left|\frac{r_1}{p_1} - \frac{r_2}{p_2}\right| = \frac{1}{p_1 p_2} |r_1 p_2 - r_2 p_1| \geq \frac{1}{4n^2},$$

since  $r_1 p_2 - r_2 p_1 \neq 0$ . Therefore, taking into account this separation between the centers and the length of the intervals, we conclude that for  $x \in G'_{p_1}$  and  $y \in G'_{p_2}$ ,

$$|x - y| \geq \frac{1}{4n^2} - \frac{2}{\mathfrak{g}(n)} \geq \frac{1}{8n^2} \quad (\text{since } \mathfrak{g}(n) \gg n^2).$$

Let  $\mathcal{P}_m^n$  be the set of all the prime numbers between  $m$  and  $n$  and define

$$H_n := \bigcup_{p \in \mathcal{P}_n^{2n}} G'_p.$$

Then  $H_n$  is the union of intervals of length at least  $\frac{2}{\mathfrak{g}(2n)}$  that are separated by a distance of at least  $\frac{1}{8n^2}$ .

Now we observe the following: If  $I$  is an interval with  $|I| > \frac{3}{n}$ , then at least  $\frac{p|I|}{3}$  of the intervals of  $G'_p$  are completely contained on  $I$ . To verify this last statement, cut  $I$  into three consecutive and congruent subintervals. Then, in the middle interval there are at least  $\frac{p|I|}{3}$  points of the form  $\frac{m}{p}$ . All the intervals of  $G'_p$  centered at these points are completely contained in  $I$ , since the length of each interval of  $G'_p$  is  $\frac{2}{\mathfrak{g}(p)} < \frac{|I|}{3}$ .

In addition, by the Prime Number Theorem, we know that  $\#(\mathcal{P}_1^n) \sim \frac{n}{\log(n)}$ , so we can find  $n_0$  such that

$$\#(\mathcal{P}_n^{2n}) \geq \frac{n}{2 \log(n)} \text{ for } n \geq n_0.$$

Hence, if  $I$  is an interval with  $|I| > \frac{3}{n}$ , then there are at least

$$\frac{p|I|}{3} \frac{n}{2 \log(n)} > \frac{n^2 |I|}{6 \log(n)}$$

intervals of  $H_n$  contained on  $I$ . Now we will construct a Cantor-type subset  $E$  of  $B_{\mathfrak{g}}$  and apply Lemma 4.3.

Consider the sequence  $\{n_k\}$  of the hypothesis of the proposition and let  $E_0 = [0, 1]$ . Define  $E_k$  as the union of all the intervals of  $H_{n_k}$  contained in  $E_{k-1}$ . Then  $E_k$  is built up of intervals of length at least  $\frac{1}{\mathfrak{g}(2n_k)}$  and separated by at least  $\varepsilon_k = \frac{1}{8n_k^2}$ . Moreover, since  $\frac{1}{\mathfrak{g}(2n_{k-1})} \geq \frac{3}{n_k}$ , each interval of  $E_{k-1}$  contains at least

$$m_k := \frac{n_k^2}{6 \log(n_k) \mathfrak{g}(2n_{k-1})}$$

intervals of  $E_k$ .

Now we can apply Lemma 4.3 to  $E = \bigcap E_k$ . Recall that  $\varepsilon_k$  denotes the separation between  $k$ -level intervals and  $m_k$  denotes the number of children of each of them. Consider  $h \prec \mathfrak{h}$ . Then

$$\begin{aligned} D_k^h &= m_1 \cdot m_2 \cdots m_{k-1} h(\varepsilon_k m_k) \\ &= \frac{6^{-(k-2)} n_2^2 \cdots n_{k-1}^2}{\log(n_2) \cdots \log(n_{k-1}) \mathfrak{g}(n_1) \cdots \mathfrak{g}(n_{k-2})} h \left( \frac{6}{\log(n_k) \mathfrak{g}(n_{k-1})} \right). \end{aligned}$$

Now we note that  $n_k \geq \log(n_k)$  and, by hypothesis (A), we also have that  $n_k \geq \mathfrak{g}(2n_{k-1}) \geq \mathfrak{g}(n_{k-1})$ . In addition,  $h$  is doubling, therefore it follows that we can bound the first factor to obtain that

$$\begin{aligned} D_k^h &\gtrsim \frac{6^{-k} n_{k-1}^2}{\log(n_{k-1}) \mathfrak{g}(n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h}) \left( \frac{1}{\log(n_k) \mathfrak{g}(n_{k-1})} \right) (\mathfrak{g}^{-1}(\log(n_k) \mathfrak{g}(n_{k-1})))^2} \\ &\gtrsim \frac{6^{-k} n_{k-1}^2}{\mathfrak{g}^2(n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h}) \left( \frac{1}{\log(n_k) \mathfrak{g}(n_{k-1})} \right) (\mathfrak{g}^{-1}(\log(n_k)) + n_{k-1})^2} \end{aligned}$$

since, by hypothesis (B),  $n_k$  satisfies  $\log(n_{k-1}) \leq \mathfrak{g}(n_{k-2})$  and  $\mathfrak{g}$  satisfies (12). Now, again by hypothesis (B),

$$D_k^h \geq \frac{1}{6^k \mathfrak{g}^2(n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h}) \left( \frac{1}{\log(n_k) \mathfrak{g}(n_{k-1})} \right)}.$$

Thus, if

$$\lim_k \frac{1}{6^k \mathfrak{g}^2(n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h}) \left( \frac{1}{\log(n_k) \mathfrak{g}(n_{k-1})} \right)} > 0,$$

then  $\mathcal{H}^h(E) > 0$  and therefore  $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$ .  $\square$

We also have an example to illustrate this last result.

**Example 4.7.** Define  $\mathfrak{g}_r(x) = e^{x^{\frac{2}{r}}}$  for  $r > 0$  and consider the set  $B_{\mathfrak{g}_r}$ . Then  $\mathfrak{h}_r(x) = \frac{1}{\log^r(\frac{1}{x})}$  will be an expected lower bound for the dimension function for the set  $B_{\mathfrak{g}_r}$ . Consider the chain  $h_\theta(x) = \frac{1}{\log^\theta(\frac{1}{x})}$  ( $0 < \theta < r$ ), which satisfy  $h_\theta \prec \mathfrak{h}_r$ . In this context,  $\Delta(h_\theta, \mathfrak{h}_r)(x) = \log^{\theta-r}(\frac{1}{x})$ . Define the sequence  $n_k$  as follows:

$$n_k = e^{kn_k^{\frac{2}{r}}}.$$

Clearly the sequence is admissible, since

(A)  $n_k \geq 3\mathfrak{g}(2n_{k-1})$ , and



$$(B) \log(n_k) \leq \mathfrak{g}(n_{k-1}).$$

Inequality (13) now becomes

$$D_k^h \gtrsim \frac{(\log \log(n_k) + n_{k-1}^{\frac{2}{r}})^{r-\theta}}{6^k e^{2n_{k-2}^{\frac{2}{r}}}} \geq \frac{n_{k-1}^{\frac{2}{r-\theta}}}{6^k e^{2n_{k-2}^{\frac{2}{r}}}}.$$

Finally, for any  $\varepsilon > 0$  and  $M > 0$ ,  $n_k$  satisfies, for large  $k$ ,

$$\frac{n_{k-1}^\varepsilon}{6^k e^{Mn_{k-1}^{\frac{2}{r}}}} = \frac{e^{\varepsilon k n_{k-1}^{\frac{2}{r}}}}{6^k e^{Mn_{k-1}^{\frac{2}{r}}}} = \frac{e^{(\varepsilon k - M)n_{k-1}^{\frac{2}{r}}}}{6^k} \geq 1,$$

so we conclude that  $\lim_k D_k^h > 0$ . Therefore the set  $E \subseteq B_{\mathfrak{g}_r}$  constructed in the proof above satisfies  $\mathcal{H}^{h_\theta}(E) > 0$  for all  $\theta < r$ .

**4.3. Another Jarník type Theorem and proof of Lemma 3.1.** For the proof of Lemma 3.1 we will need a different but essentially equivalent formulation of Jarník's theorem. We first recall that in the proof of Theorem 2.4 we have used the following theorem (see for example [Fal86])

**Theorem 4.8.** *Let  $\mathfrak{n} = \{n_j\}_j$  be a increasing sequence with  $n_{j+1} \geq n_j^j$  for all  $j \in \mathbb{N}$ . For  $0 < \alpha \leq 1$ , if  $A_\alpha^n$  is defined as*

$$A_\alpha^n = \left\{ x \in [0, 1] \setminus \mathbb{Q} : \forall j \exists p, q ; q \leq n_j^\alpha ; |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\},$$

*then  $\dim_H(A_\alpha^n) = \alpha$ .*

For the proofs of Theorem 4.8 and Theorem 4.5, we refer the reader to [Jar31], [Bes34], [Egg52], [Fal86], and [Fal03].

Now we want to relate the sets  $A_\alpha$  and  $B_\beta$  and their generalized versions. It is clear that for any  $\alpha \in (0, 1]$ , we have the inclusion  $A_\alpha \subset B_{\frac{2}{\alpha}}$ . For  $\alpha \in (0, 1]$ , if  $x \in A_\alpha$  then for each  $j \in \mathbb{N}$  there exists a rational  $\frac{p_j}{q_j}$  with  $q_j \leq n_j^\alpha$  such that  $|x - \frac{p_j}{q_j}| < n_j^{-2}$ , which is equivalent to  $|xq_j - p_j| < q_j n_j^{-2}$ . Therefore  $|xq_j - p_j| \leq q_j^{\frac{1-\alpha}{\alpha}}$ . Observe that if there were only finite values of  $q$  for a given  $x$ , then  $x$  has to be rational. For if  $q_j = q_{j_0}$  for all  $j \geq j_0$ , then  $|x - \frac{p_j}{q_{j_0}}| \rightarrow 0$  and this implies that  $x \in \mathbb{Q}$ . We conclude then that, for any  $x \in A_\alpha$ ,  $\|xq\| < \frac{1}{q^{\frac{1}{\alpha}-1}}$  for infinite many  $q$  and therefore  $x \in B_{\frac{2}{\alpha}}$ . However, since the dimension of  $A_\alpha^n$  coincides with the one of  $B_{\frac{2}{\alpha}}$ , one can expect that both sets have approximately comparable sizes.

We introduce the following definition, which is the extended version of the definition of the set  $A_\alpha^n$  in Theorem 4.8.

**Definition 4.9.** Let  $\mathfrak{n} = \{n_j\}_j$  be any increasing nonnegative sequence of integers. Let  $\mathfrak{f}$  be a increasing function defined on  $\mathbb{R}_+$ . Define the set

$$A_\mathfrak{f}^{\mathfrak{n}} := \left\{ x \in [0, 1] \setminus \mathbb{Q} : \forall j \exists p, q ; q \leq \mathfrak{f}(n_j) ; |x - \frac{p}{q}| < \frac{1}{n_j^2} \right\}.$$

The preceding observation about the inclusion  $A_\alpha \subset B_\beta$  can be extended to this general setting. For a given  $\mathfrak{g}$  as in the definition of  $B_\mathfrak{g}$ , define  $\Gamma_\mathfrak{g}(x) = \mathfrak{g}^{-1}(x^2)$ . Then the same calculations show that  $A_{\Gamma_\mathfrak{g}}^\mathfrak{n} \subset B_\mathfrak{g}$ .

We will need a converse relation between those sets, since we want to prove a lower bound for the sets  $A_\mathfrak{f}^\mathfrak{n}$  from the estimates provided in Proposition 4.6.

**Lemma 4.10.** *Let  $\mathfrak{g}$  and  $B_\mathfrak{g}$  be as in Proposition 4.6. Define  $\Gamma_\mathfrak{g}(x) = \mathfrak{g}^{-1}(x^2)$ . Then, if  $\mathfrak{m} = \{m_k\}$  is the sequence defining the set  $E$  in the proof of Proposition 4.6, then the set  $E$  is contained in  $A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$ , where  $\mathfrak{n} = \{n_k\} = \{\mathfrak{g}(m_k)^{\frac{1}{2}}\}$ .*

*Proof.* Recall that in the proof of Proposition 4.6 we define the sets  $G'_q$  as a union of intervals of the form  $I = \left(\frac{r}{q} - \frac{1}{\mathfrak{g}(q)}; \frac{r}{q} + \frac{1}{\mathfrak{g}(q)}\right)$ . The sets  $H_n$  were defined as  $H_n := \bigcup_{p \in \mathcal{P}_n^{2n}} G'_p$ , where  $\mathcal{P}_n^{2n}$  is the set of primes between  $n$  and  $2n$ . We can therefore write

$$H_n := \bigcup I_j^n.$$

Now, given a sequence  $\mathfrak{m} = \{m_k\}$ , for each  $k$ , the set  $E_k$  is defined as the union of all the intervals of  $H_{m_k}$  that belong to  $E_{k-1}$ , where  $E_0 = [0, 1]$ . If  $E = \bigcap E_k$ , any  $x \in E$  is in  $E_k$  and therefore in some of the  $I_j^{m_k}$ . It follows that there exists integers  $r$  and  $q$ ,  $q \leq 2m_k$  such that

$$\left|x - \frac{r}{q}\right| < \frac{1}{\mathfrak{g}(q)} < \frac{1}{\mathfrak{g}(m_k)} = \frac{1}{n_k^2}, \quad q \leq 2\mathfrak{g}^{-1}(n_k^2).$$

Therefore  $E \subset A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$ . □

We remark that the above inclusion implies that any lower estimate on the size of  $E$  would also be a lower estimate for  $A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$ .

We now conclude the proof of Theorem 3.2 by proving Lemma 3.1:

*Proof of Lemma 3.1:* Let  $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$ . For  $r > 1$ , consider the function  $\mathfrak{g}_r$ , the sequence  $\mathfrak{m} = \{m_k\}$  and the set  $E_r$  as in Example 4.7. Define  $\mathfrak{f} = \Gamma_{\mathfrak{g}_r}$ ,  $\mathfrak{n}$  and  $A_\mathfrak{f}^\mathfrak{n}$  as in Lemma 4.10. It follows that  $\mathfrak{f}(x) = \log(x^2)^{\frac{r}{2}}$ ,  $n_j = e^{\frac{1}{2}n_j^{\frac{4}{r}-1}}$  and

$$A_\mathfrak{f}^\mathfrak{n} := \left\{x \in [0, 1] \setminus \mathbb{Q} : \forall j \exists p, q ; q \leq \mathfrak{f}(n_j); \left|x - \frac{p}{q}\right| < \frac{1}{n_j^2}\right\}.$$

Note that Lemma 4.10 says that the inclusion  $E \subset A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$  *always* holds, for any defining function  $\mathfrak{g}$ , where  $E$  is the *substantial portion* of the set  $B_\mathfrak{g}$  (see Proposition 4.6). But we need the positivity of  $\mathcal{H}^b(E)$  to conclude that the set  $A_{\Gamma_\mathfrak{g}}^\mathfrak{n}$  also has positive  $\mathcal{H}^b$ -measure. For the precise choices of  $\mathfrak{h}$  and  $\mathfrak{g}$ , we obtain this last property from Example 4.7. Precisely,  $\mathcal{H}^b(E_r) > 0$  and therefore the set  $A_\mathfrak{f}^\mathfrak{n}$  has positive  $\mathcal{H}^b$ -measure. This concludes the proof of Lemma 3.1 and therefore the set constructed in the proof of Theorem 3.2 fulfills the condition of being an  $F_\mathfrak{h}$ -set. □

## ACKNOWLEDGMENTS

We would like to thank to Pablo Shmerkin for many interesting suggestions and remarks about this problem.

## REFERENCES

- [Bes34] A. S. Besicovitch, *Sets of fractional dimensions IV: On rational approximation to real numbers.*, J. London Math. Soc. **9** (1934), 126–131.
- [CHM10] Carlos A. Cabrelli, Kathryn E. Hare, and Ursula M. Molter, *Classifying Cantor sets by their fractal dimensions*, Proc. Amer. Math. Soc. **138** (2010), no. 11, 3965–3974.
- [Egg52] H. G. Eggleston, *Sets of fractional dimensions which occur in some problems of number theory*, Proc. London Math. Soc. (2) **54** (1952), 42–93.
- [EK06] Márton Elekes and Tamás Keleti, *Borel sets which are null or non- $\sigma$ -finite for every translation invariant measure*, Adv. Math. **201** (2006), no. 1, 102–115.
- [Fal86] K. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986.
- [Fal03] Kenneth Falconer, *Fractal geometry*, second ed., John Wiley & Sons Inc., Hoboken, NJ, 2003, Mathematical foundations and applications.
- [Hau18] Felix Hausdorff, *Dimension und äußeres Maß*, Math. Ann. **79** (1918), no. 1-2, 157–179.
- [Jar31] Vojtěch Jarník, *Über die simultanen diophantischen Approximationen*, Math. Z. **33** (1931), no. 1, 505–543.
- [Khi24] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92** (1924), no. 1-2, 115–125.
- [KN74] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience [John Wiley & Sons], New York, 1974, Pure and Applied Mathematics.
- [KT01] Nets Katz and Terence Tao, *Some connections between Falconer’s distance set conjecture and sets of Furstenberg type*, New York J. Math. **7** (2001), 149–187 (electronic).
- [MR10] Ursula Molter and Ezequiel Rela, *Improving dimension estimates for Furstenberg-type sets*, Adv. Math. **223** (2010), no. 2, 672–688.
- [OR06] L. Olsen and Dave L. Renfro, *On the exact Hausdorff dimension of the set of Liouville numbers. II*, Manuscripta Math. **119** (2006), no. 2, 217–224.
- [Rog98] C. A. Rogers, *Hausdorff measures*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1970 original, With a foreword by K. J. Falconer.
- [Shm11] Pablo Shmerkin, *Private communication*.
- [Wol99] Thomas Wolff, *Recent work connected with the Kakeya problem*, Prospects in mathematics (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, 1999, pp. 129–162.

DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES,  
UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, PABELLÓN I, 1428 CAPITAL  
FEDERAL, ARGENTINA, AND IMAS - CONICET, ARGENTINA

*E-mail address*, Ursula Molter: `umolter@dm.uba.ar`

*E-mail address*, Ezequiel Rela: `erela@dm.uba.ar`